Some important states of homogeneous strain

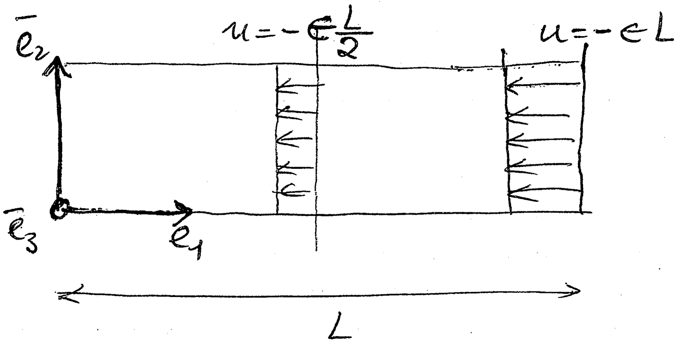
(I) Uniaxial compression

Uniaxial compession in the $\bar{e}_1$ direction is defined by the displacement field:

$$\bar{u} = -\epsilon X_1 \bar{e}_1 \quad ; \quad u_1 = -\epsilon X_1 , \quad u_2 = u_3 = 0$$

$$\epsilon = \text{const}$$

$$[\epsilon] = \begin{bmatrix} -\epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Note: $\bar{u}$ varies linearly with X_1, but the resulting strain is uniform throughout the body.

(II) Simple Shear

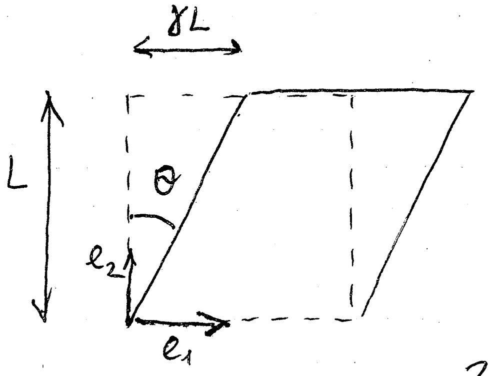
Simple shear with respect to $(\bar{e}_1, \bar{e}_2)$ is defined by

$$\bar{u} = \gamma X_2 \bar{e}_1 \quad : \quad u_1 = \gamma X_2 , \quad u_2 = u_3 = 0$$

$$[\epsilon] = \begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The non-zero engineering shear strain is

$$\gamma_{12} = 2 \cdot \epsilon_{12} = \gamma = \tan \theta$$



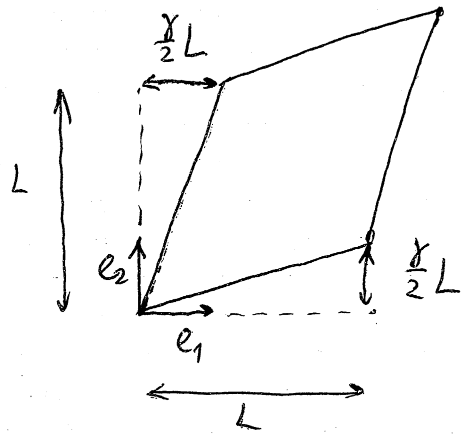
III Pure shear

pure shear with respect to X_1, X_2 is difined by

$$\bar{u} = \frac{\gamma}{2} X_2 \bar{e}_1 + \frac{\gamma}{2} X_1 \bar{e}_2 \; ; \quad u_1 = \frac{\gamma}{2} X_2 \; , \quad u_2 = \frac{\gamma}{2} X_1 \; , \quad u_3 = 0$$

The matrix of the components of $\bar{\bar{\epsilon}}$ is

$$[\bar{\bar{\epsilon}}] = \begin{bmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$\gamma > 0$

$$\gamma_{12} = 2 \cdot \epsilon_{12} = \gamma$$

While the simple shear & pure shear strain tensors are the same ; the rotation is not the same.

While for pure shear $\bar{\bar{\omega}}^{PS} = \bar{\bar{0}}$ for simple shear:

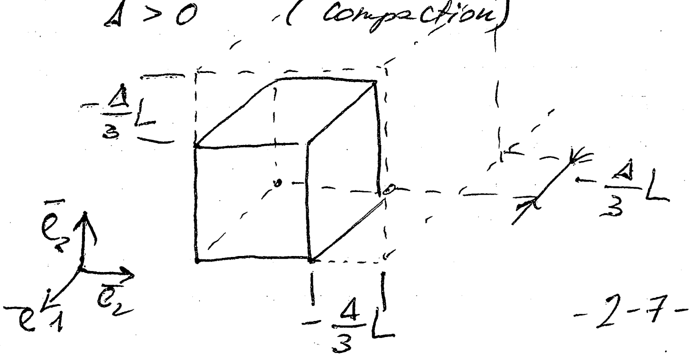
$$[\bar{\bar{\omega}}^{SS}] = \begin{pmatrix} 0 & \gamma/2 & 0 \\ -\gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

IV Uniform dilatation (compaction)

The displacement field in uniform compaction / dilitation is given by

$$u = -\left(\frac{1}{3}\Delta\right) X_1 \bar{e}_1 - \left(\frac{1}{3}\Delta\right) X_2 \bar{e}_2 - \left(\frac{1}{3}\Delta\right) X_3 \bar{e}_3 \; ,$$

$\Delta > 0$ (compaction)



$$[\bar{\bar{\epsilon}}] = \begin{pmatrix} -\frac{\Delta}{3} & 0 & 0 \\ 0 & -\frac{\Delta}{3} & 0 \\ 0 & 0 & -\frac{\Delta}{3} \end{pmatrix}$$

Note that in uniform dilation the volume change is

$$\frac{V - V_0}{V_0} = (1 + \epsilon_{11})(1+\epsilon_{22})(1+\epsilon_{33}) - 1$$

Small volume element:

$$V_0 = dX_1\, dX_2\, dX_3 \quad ; \quad V = \left(dX_1 + \frac{\partial u_1}{\partial X_1} dX_1\right)\left(dX_2 + \frac{\partial u_2}{\partial X_2} dX_2\right)\left(dX_3 + \frac{\partial u_3}{\partial X_3} dX_3\right)$$

$$\frac{V - V_0}{V_0} = \left(1 + \frac{\partial u_1}{\partial X_1}\right)\left(1 + \frac{\partial u_2}{\partial X_2}\right)\left(1 + \frac{\partial u_3}{\partial X_3}\right) - 1 = (1+\epsilon_{11})(1+\epsilon_{22})(1+\epsilon_{33}) - 1$$

$$\frac{V - V_0}{V_0} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} + O(\epsilon)^2 \qquad |\epsilon_{ij}| \ll 1$$

$$\frac{V - V_0}{V_0} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \sum_K \epsilon_{KK} = tr[\bar{\bar{\epsilon}}]$$

Note : that it holds for any combination of ϵ_{ij}

<u>Nonlinear strain tensor</u>:

<u>Remarks</u>:

1) Note the strain components are <u>nondimentional</u>
2) The assuption of small deformation $|u_{i,j}| \ll 1$ was used widely

$|\epsilon_{ij}| \sim 10^{-4}$ small

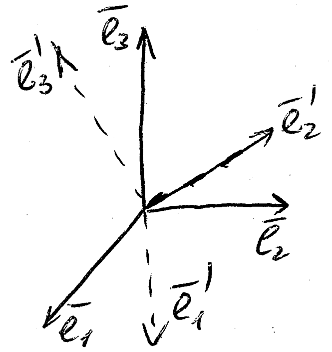
$|\epsilon_{ij}| \sim 10^{-3}$ in the trancition range of small strains

$|\epsilon_{ij}| \sim 10^{-2}$ (few percent) – we are reaching large strains

In case of large strain : a Cauchy - Green strain tensor is usually used.
This is not in the scope of this cource.

$$\epsilon_{ij}^{nl} = \epsilon_{ij} + \frac{1}{2}\left(\sum u_{K,i}\, u_{K,j} \right)$$

Transformation of strain tensor.



How the components of strain tensor ϵ_{ij} will change in the new coordinate system :

$$\epsilon'_{ij} = Q_{ki}\, Q_{mj}\, \epsilon_{km}$$

$$\bar{\bar{\epsilon}}' = \bar{\bar{Q}}^T \bar{\bar{\epsilon}}\, \bar{\bar{Q}}$$

$$\bar{\bar{\epsilon}} = \bar{\bar{Q}}\, \bar{\bar{\epsilon}}'\, \bar{\bar{Q}}^T$$

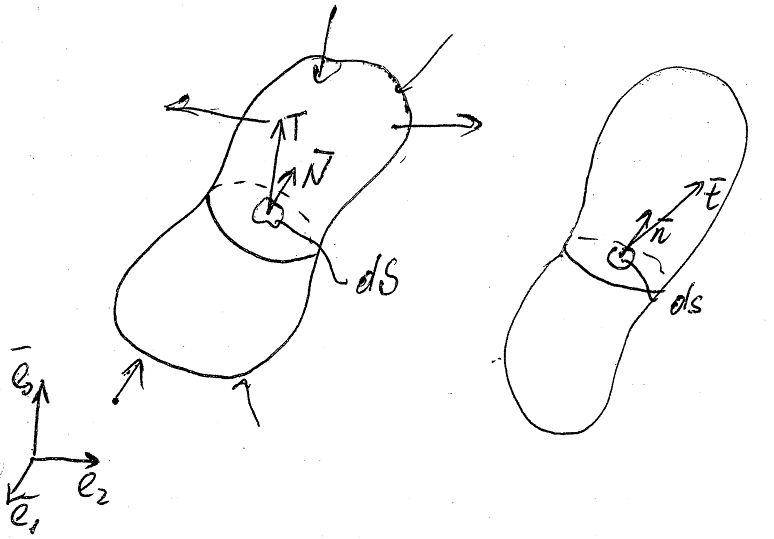
Remind:

$\bar{\bar{Q}}$ is the orthogonal matrix or transformation matrix

$$\bar{e}'_i = \bar{\bar{Q}}\, \bar{e}_i = Q_{ji}\, \bar{e}_j$$

Concept of Stress.

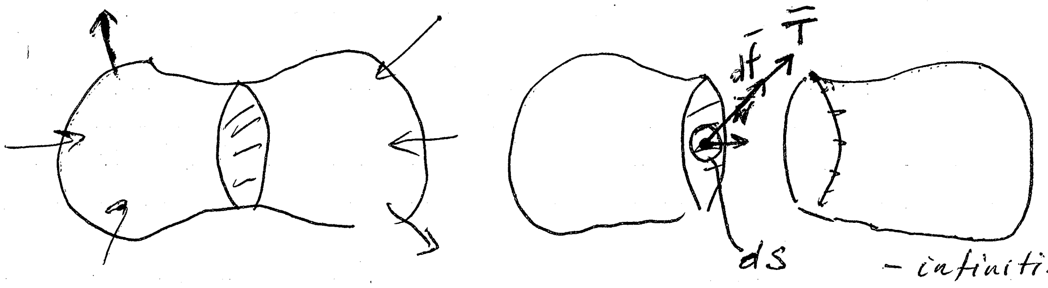
Traction Vectors, and Stress tensors.



Consider a body under action of some external forces.

Let the body now be cut (imaginary) by a plane surface.

Obviously there are forces acting on that surface, to keep the body in equilibrium.



$\vec{N}$ is a unit normal victor to that cutting plane. surface.

dS – infinitisimal surface element

There will be a force resultant $\bar{T}$ acting on that infinitisinal surface element.

$$T = \frac{df}{dS} \qquad t = \frac{df}{ds}$$

$d\bar{f} = \bar{t}\,ds = \bar{T}\,dS$; assume (as ussuall) that the deformations are small & there is NO change in dS, so $dS = ds$ $(|\epsilon_{i,j}| \ll 1)$

Then $T = t$

We can also write

$$\boxed{\bar{t}^{(\vec{x},\vec{n})} = \lim_{S \to 0} \frac{\Delta \bar{f}}{\Delta S} = \frac{d\bar{f}}{dS}}$$

We defined <u>surface traction vector</u>. $\bar{t}(\bar{x}, \bar{n})$

- the traction vector $\bar{t}(\bar{x}, \bar{n})$ denotes force per unit area
- $\bar{t}(\bar{x}, \bar{n})$ is colinear with $d\bar{f}$, but <u>in general</u> is not in the same direction as $\bar{n}$
- $\bar{t}(\bar{x}, \bar{n})$ depends on the position of $\bar{x}$, and indeed on the plane chosen – defined by $\bar{n}$ – (there is an infinity of planes that pass through $\bar{x}$)

Components of stress at a point.

Cauchy's stress theorem.

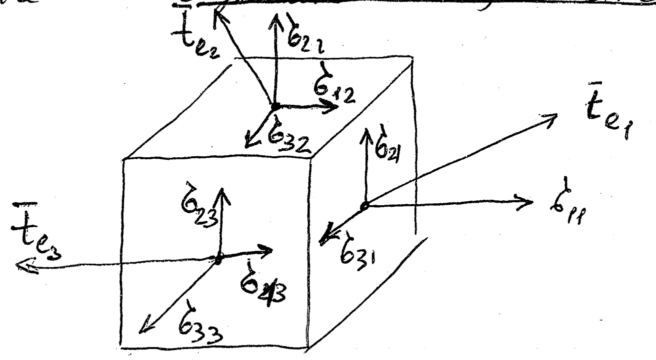
$$\boxed{\bar{t}(\bar{x},\bar{n}) = \bar{\bar{\sigma}}(\bar{x})\,\bar{n}}$$

an alternative $\bar{T}(\bar{X},\bar{N}) = \bar{\bar{P}}(\bar{X})\bar{N}$

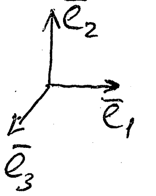
$\bar{\bar{\sigma}}$ is Cauchy stress (true stress) ; $\bar{\bar{P}}$ - first Piola-Kirchhoff stress (~~engineering~~ nominal stress)

$$t_i = \sigma_{ij}\, n_j$$

Let's ~~prove~~ derive the theorem; but before that let's define ~~components~~ components of stress tensor



$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

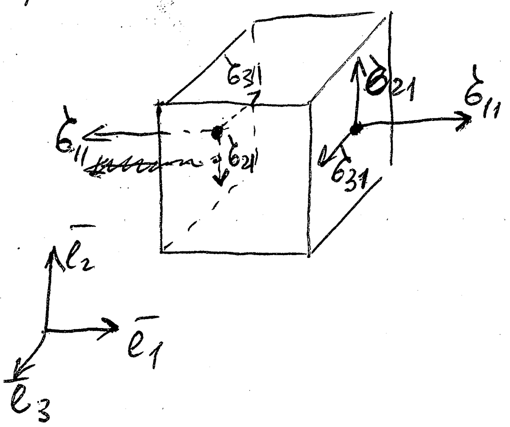


$$\sigma_{11} \equiv \bar{t}_{e_1}\cdot\bar{e}_1 \;;\quad \sigma_{12} \equiv \bar{t}_{e_2}\cdot\bar{e}_1 \;;\quad \sigma_{13} = \bar{t}_{e_3}\cdot\bar{e}_1$$
$$\sigma_{21} \equiv \bar{t}_{e_1}\cdot\bar{e}_2 \;;\quad \sigma_{22} \equiv \bar{t}_{e_2}\cdot\bar{e}_2 \;;\quad \sigma_{23} = \bar{t}_{e_3}\cdot\bar{e}_2$$
$$\sigma_{31} \equiv \bar{t}_{e_1}\cdot\bar{e}_3 \;;\quad \sigma_{23} \equiv \bar{t}_{e_2}\cdot\bar{e}_3 \;;\quad \sigma_{33} = \bar{t}_{e_3}\cdot\bar{e}_3$$

- $\{\sigma_{11}, \sigma_{22}, \sigma_{33}\}$ are called normal stresses
- $\{\sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{32}, \sigma_{21}, \sigma_{31}\}$ shear stress components

Sign convention for positive normal & shear stress

Normal stress is considered as "positive" when it produces tension ; "negative" — compression



The positive direction of a shear component on any face is taken in the positive (negative) direction of the coordinate axis if the positive normal stress of the same face is in the positive (negative) direction of the corresponding axis.

Plane stress

In many examples some components of stress vanish. One specific case is called **plane stress**. This is a common assumption in case of thin structures (thin plates).

- There are no tractions acting on $\pm \bar{e}_3$ faces, so

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$$

- There is no $\bar{e}_3$-components of tractions acting on $\pm \bar{e}_1$ and $\pm \bar{e}_2$ so:

$$\sigma_{31} = \sigma_{32} = 0$$

Thus, the components of stress tensor become

$$[\sigma] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>traction vector $\bar{t}$ on a surface with normal $\bar{n}$ in terms of stress $\bar{\sigma}$.</u>

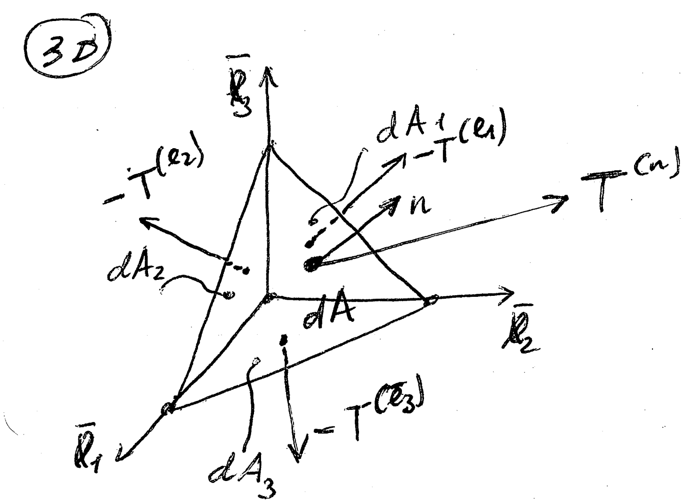
1. <u>2-D : plane stress result.</u>

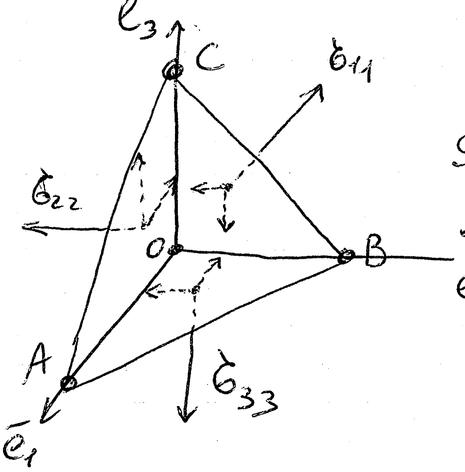
(d) Cauchy stress theorem.

Discuss the need in $[\sigma_{ij}]$ $\updownarrow$ t_i

Consider tetrahedron

(3D)





$S_{OBC} = S_{ABC} \cdot \underbrace{\cos(\bar{n},\bar{e}_1)}_{n_1}$

$S_{AOC} = S_{ABC} \cdot \underbrace{\cos(\bar{n},\bar{e}_2)}_{n_2}$

$S_{AOB} = S_{ABC} \cdot \underbrace{\cos(\bar{n},\bar{e}_3)}_{n_3}$

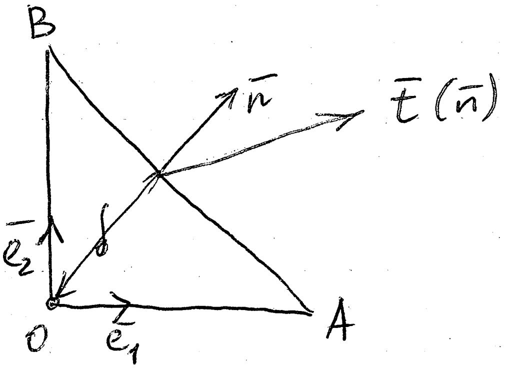
Equilibrium in $\bar{e}_1$:

$$-\sigma_{11} S_{OBC} - \sigma_{13} S_{AOB} - \sigma_{12} S_{AOC} + t_1 S_{ABC} + b_1 \cdot \underbrace{\tfrac{1}{3} h S_{ABC}}_{V\text{ - volume}\ \to 0,\ h \to 0} = 0$$

$$-\sigma_{11} n_1 - \sigma_{13} n_3 - \sigma_{12} n_2 + t_1 = 0$$

(2D) For simplicity, we will consider a state of plane stress with a body force $\bar{b}$ (per unit volume) $b_i = (b_1, b_2, 0)$

Triangular prism with $\Delta X_3 = 1$ (unit thickness in the $\bar{e}_3$ - direction)

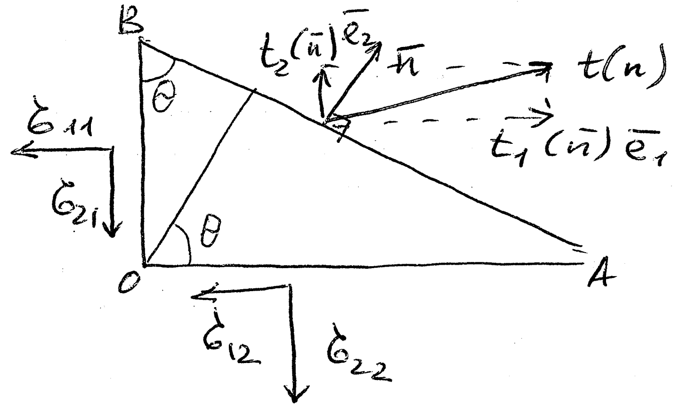


δ is the distance between $\overline{AB}$ & $\bar{O}$

The prisme is assumed to be in equilibrium under the actions of the tractions (on the 3 rectangular faces of the prism) & under the action of the body force (b_1, b_2).

<u>Note</u>: the outward normals to the faces in the $(\bar{e}_2, \bar{e}_3)$-plane & in the $(\bar{e}_1, \bar{e}_3)$-plane, are in the <u>negative</u> directions of $\bar{e}_1$ & $\bar{e}_2$, respectively.

Therefore, the components of the traction vectors on these faces are $(-\sigma_{11}, -\sigma_{21})$ & $(-\sigma_{22}, -\sigma_{12})$



$$t(n) = t_1(\bar{n})\,\bar{e}_1 + t_2(\bar{n})\,\bar{e}_2$$

Now let us sum forces in the $\bar{e}_1$ & $\bar{e}_2$-directions

$(\Delta X_3 = 1)$

[Remember $\bar{t}$ - is traction with units $\frac{\text{force}}{\text{area}}$]

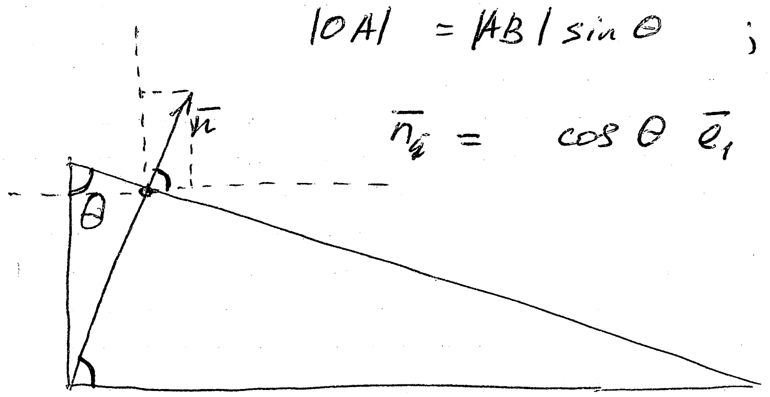
$$\bar{e}_1 : \overset{1}{(\Delta X_3)}\Big[\, t_1(\bar{n})\,|AB| - \sigma_{11}\, OB - \sigma_{12}\, OA - \sigma_1 \left(\tfrac{1}{2} AB \cdot \delta\right)\Big] = 0$$

$$\bar{e}_2 : \overset{1}{(\Delta X_3)}\Big[\, t_2(\bar{n})\,|AB| - \sigma_{22}\, OA - \sigma_{21}\, OB - \sigma_2 \left(\tfrac{1}{2} AB \cdot \delta\right)\Big] = 0$$

{ Note : $\sin\theta = \dfrac{|OA|}{|AB|}$, $\cos\theta = \dfrac{|OB|}{|AB|}$

$|OA| = |AB|\sin\theta$; $OB = |AB|\cos\theta$

$\bar{n} = \cos\theta\,\bar{e}_1 + \sin\theta\,\bar{e}_2$



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with $\delta \to 0$

$$\begin{cases} \bar{e}_1 : & t_1(\bar{n}) - \sigma_{11}\cos\theta - \sigma_{12}\sin\theta = 0 \\ \bar{e}_2 : & t_2(\bar{n}) - \sigma_{21}\cos\theta - \sigma_{22}\sin\theta = 0 \end{cases}$$

$$\begin{pmatrix} t_1(\bar{n}) \\ t_2(\bar{n}) \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Can be extended to 3D state by considering a tetrahedron instead.

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \quad ; \quad t_i = \sigma_{ij} n_j \quad (3.1)$$

⇒ Thus, at any point in the body the set of nine quantities σ_{ij} provide a complete picture of the state of stress at that point.

⇒ Given the components of stress σ_{ij} at a point, the components of the traction vector t_i on an arbitrary surface element (that point outward with normal components n_j) can be obtained by using Eq. (3-1).

Hydrostatic pressure state

$$\sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

let's calculate the traction vector t_i acting on an arbitrary plane $\bar{n}$.

$$t_i = \sigma_{ij} n_j \quad ; \quad \sigma_{ij} = -p\,\delta_{ij}$$

$$t_i = -p\,\delta_{ij} n_j = -p\, n_i \qquad \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} -p\,n_1 \\ -p\,n_2 \\ -p\,n_3 \end{Bmatrix}$$

$\bar{t} = -p\vec{n}$; so the traction vector will be always in the direction $\bar{n}$, and will take the value $(-p)$

Note: it does not matter at which angle we cut / put our surface.